(1) Consider the nonlinear oscillator described by the Hamiltonian

\[ H(q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 + \frac{1}{4}\epsilon aq^4 + \frac{1}{4}\epsilon b p^4 , \]

where \( \epsilon \) is small.

(a) Find the perturbed frequencies \( \nu(J) \) to lowest nontrivial order in \( \epsilon \).

(b) Find the perturbed frequencies \( \nu(A) \) to lowest nontrivial order in \( \epsilon \), where \( A \) is the amplitude of the \( q \) motion.

(c) Find the relationships \( \phi = \phi(\phi_0, J_0) \) and \( J = J(\phi_0, J_0) \) to lowest nontrivial order in \( \epsilon \).

Solution:

With \( k \equiv m \nu_0^2 \), recall the AA variables

\[ \phi_0 = \tan^{-1}\left( \frac{\nu_0 q}{p} \right) , \quad J_0 = \frac{p^2}{2m} + \frac{1}{2}m \nu_0^2 q^2 . \]

Thus, \( q = (2J_0/m\nu_0)^{1/2} \sin \phi_0 \) and \( p = (2m\nu_0 J_0)^{1/2} \cos \phi_0 \), so the Hamiltonian is

\[ \tilde{H}(\phi_0, J_0) = \nu_0 J_0 + \epsilon \tilde{H}_1(\phi_0, J_0) , \]

where

\[ \tilde{H}_1(\phi_0, J_0) = \frac{a J_0^2}{m^2 \nu_0^2} \sin^4 \phi_0 + b m^2 \nu_0^2 J_0^2 \cos^4 \phi_0 . \]

(a) Averaging over \( \phi_0 \), we have \( \langle \sin^4 \phi_0 \rangle = \langle \cos^4 \phi_0 \rangle = \frac{3}{8} \), so

\[ E_1(J) = \langle \tilde{H}_1(\phi_0, J) \rangle = \left( \frac{a}{mk} + bmk \right) \times \frac{3}{8} J^2 . \]

The perturbed frequencies are \( \nu(J) = \nu_0 + \epsilon \nu_1 \) where \( \nu_1 = \frac{\partial E_1}{\partial J} \). Thus,

\[ \nu(J) = \sqrt{\frac{k}{m}} + \left( \frac{a}{mk} + bmk \right) \times \frac{3}{8} \epsilon J^2 . \]

(b) We only need \( J \) to zeroth order in \( \epsilon \). Setting \( p = 0 \) and \( q = A \) gives \( J = \frac{1}{2}kA^2 + \mathcal{O}(\epsilon) \), in which case

\[ \nu(A) = \sqrt{\frac{k}{m}} + \left( \frac{a}{mk} + bmk \right) \times \frac{3}{8} \epsilon A^2 . \]
(c) Recall the desired type-II CT is generated by $S(\phi_0, J) = \phi_0 J + \epsilon S_1(\phi_0, J) + \ldots$, with

$$\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \tilde{H}_1 \rangle - \tilde{H}_1}{\nu_0(J)}.$$ 

Thus,

$$\frac{\partial S_1}{\partial \phi_0} = \frac{aJ^2}{m^2\nu_0^2} \left( \frac{3}{8} - \sin^4 \phi_0 \right) + bm^2\nu_0 J \left( \frac{3}{8} - \cos^4 \phi_0 \right).$$

Integrating, we have

$$S_1(\phi_0, J) = \frac{aJ^2}{m^2\nu_0^2} \left( \frac{1}{4} \sin(2\phi_0) - \frac{1}{32} \sin(4\phi_0) \right) - bm^2\nu_0 J^2 \left( \frac{1}{4} \sin(2\phi_0) + \frac{1}{32} \sin(4\phi_0) \right).$$

The constant may be set to zero as it leads to a constant shift of the angle variable $\phi$. Thus, we have

$$J_0 = J + \epsilon \frac{\partial S_1}{\partial \phi_0} + O(\epsilon^2)$$

$$= J + \left( \frac{a - bm^4\nu_0^4}{2m^2\nu_0^3} \right) \epsilon J^2 \cos(2\phi_0) - \left( \frac{a + bm^2\nu_0^4}{8m^2\nu_0^2} \right) \epsilon J^2 \cos(4\phi_0) + O(\epsilon^2).$$

Thus,

$$J = J_0 - \left( \frac{a - bm^4\nu_0^4}{2m^2\nu_0^3} \right) \epsilon J_0^2 \cos(2\phi_0) + \left( \frac{a + bm^2\nu_0^4}{8m^2\nu_0^2} \right) \epsilon J_0^2 \cos(4\phi_0) + O(\epsilon^2).$$

We then have

$$\phi = \phi_0 + \epsilon \frac{\partial S_1}{\partial J} + O(\epsilon^2)$$

$$= \phi_0 + \left( \frac{a - bm^4\nu_0^4}{2m^2\nu_0^3} \right) \epsilon J_0 \sin(2\phi_0) - \left( \frac{a + bm^2\nu_0^4}{16m^2\nu_0^2} \right) \epsilon J_0 \sin(4\phi_0) + O(\epsilon^2).$$

(2) Consider the forced modified van der Pol equation,

$$\ddot{x} + \epsilon(x^4 - 1) \dot{x} + x = \epsilon f_0 \cos(t + \epsilon \nu t),$$

where $\epsilon$ is small. Carry out the multiple scale analysis to order $\epsilon$. Following §3.3.2 in the Lecture Notes, find and analyze the equation which relates the amplitude $A$, detuning $\nu$, and force amplitude $f_0$ for entrained oscillations. Perform the requisite linear stability analysis and make a plot similar to that in Fig. 3.4 of the Lecture Notes. Is there a region of entrained oscillations which exhibits hysteresis as the detuning parameter is varied? If so, find the corresponding range of $f_0$ over which this occurs.

**Bonus:** Use Mathematica or Matlab to integrate the equation, showing examples of entrained and heterodyne behavior, as in Fig. 3.6 (1000 Quatloos extra credit).
Solution:

In the multiple scale analysis (MSA), we define a hierarchy of time scales \( T_n = \epsilon^n t \), and we expand \( x(t) = \sum_{n=0}^{\infty} \epsilon^n x(nT_0, T_1, \ldots) \). The general forced nonlinear oscillator equation is written

\[
\ddot{x} + x = \epsilon h(x, \dot{x}) + \epsilon f_0 \cos(t + \epsilon \nu t)
\]

where \( \epsilon \nu \) is the detuning. We write \( \frac{d}{dt} = \sum_{k=0}^{\infty} \epsilon^k \frac{\partial}{\partial T_k} \) and derive a hierarchy order by order in \( \epsilon \). As shown in §3.3 of the Lecture Notes, to lowest order we have

\[
\left( \frac{\partial^2}{\partial T_0^2} + 1 \right) x_0 = 0 \quad \Rightarrow \quad x_0 = A \cos(T_0 + \phi)
\]

where the amplitude \( A = A(T_1, T_2, \ldots) \) and phase \( \phi = \phi(T_1, T_2, \ldots) \) are independent of \( T_0 \).

At the next level of the hierarchy, we define \( \theta = T_0 + \psi_1 \) and \( \psi(T_1) \equiv \phi(T_1) - \nu T_1 \), where dependences on the scales \( \{T_1, T_3, \ldots\} \) are implicit. At order \( \epsilon^1 \), we have

\[
\left( \frac{\partial^2}{\partial \theta^2} + 1 \right) x_1 = 2 \frac{\partial A}{\partial T_1} \sin \theta + 2A \frac{\partial \phi}{\partial T_1} \cos \theta + h(A \cos \theta, -A \sin \theta) + f_0 \cos(\theta - \psi)
\]

We Fourier transform the function \( h(A \cos \theta, -A \sin \theta) \), writing

\[
h(A \cos \theta, -A \sin \theta) = \sum_{k=0}^{\infty} \left[ \alpha_k(A) \sin(k\theta) + \beta_k(A) \cos(k\theta) \right]
\]

We then have

\[
\left( \frac{\partial^2}{\partial \theta^2} + 1 \right) x_1 = \sum_{k \neq 1} \left[ \alpha_k(A) \sin(k\theta) + \beta_k(A) \cos(k\theta) \right]
\]

where the secular forcing \( k = 1 \) terms are eliminated by the requirements

\[
\frac{dA}{dT_1} = -\frac{1}{2} \alpha_1(A) - \frac{1}{2} f_0 \sin \psi
\]

\[
\frac{d\psi}{dT_1} = -\nu - \frac{\beta_1(A)}{2A} - \frac{f_0}{2A} \cos \psi
\]

which may be written as coupled ODEs since the time scales \( \{T_2, T_3, \ldots\} \) do not appear.

At any fixed point, then, one must have

\[
\left[ \alpha_1(A) \right]^2 + \left[ 2 \nu A + \beta_1(A) \right]^2 = f_0^2
\]

The linearized map in the vicinity of the fixed point \( (A^*, \psi^*) \) is given by

\[
\frac{d}{dT_1} \begin{pmatrix} \delta A \\ \delta \psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \alpha_1'(A) & \nu A + \frac{1}{2} \beta_1(A) \\ -\frac{\beta_1'(A)}{2A} - \frac{\nu}{A} & -\frac{\alpha_1(A)}{2A} \end{pmatrix} \begin{pmatrix} \delta A \\ \delta \psi \end{pmatrix}
\]

\[3\]
In our case, \( h(x, \dot{x}) = (1 - x^4) \dot{x} \), and therefore
\[
h(A \cos \theta, -A \sin \theta) = (1 - A^4 \cos^4 \theta)(-A \sin \theta)
= \left(\frac{A^5}{8} - A\right) \sin \theta + \frac{3}{16} A^5 \sin(3\theta) + \frac{1}{16} A^5 \sin(5\theta) \quad .
\]

Thus,
\[
\alpha_1(A) = \frac{1}{8} A^5 - A \quad , \quad \alpha_3(A) = \frac{3}{16} A^5 \quad , \quad \alpha_5(A) = \frac{1}{16} A^5 \quad ,
\]
where all other \( \alpha_k(A) = 0 \) and all \( \beta_k(A) = 0 \). In particular, \( \beta_1(A) = 0 \), hence
\[
G(y) \equiv \frac{1}{64} y^5 - \frac{1}{4} y^3 + (1 + 4\nu^2) y = f_0^2 \quad ,
\]
where \( y = A^2 \); note that \( G(0) = 0 \). We must analyze the behavior of \( G(y) \) for \( y \geq 0 \). Taking the derivative,
\[
G'(y) = \frac{5}{64} y^4 - \frac{3}{4} y^2 + (1 + 4\nu^2) \quad .
\]
The roots \( G'(y) = 0 \) lie at \( y = y_\pm \), where
\[
y_\pm^2 = \frac{8}{5} \left(3 \pm 2\sqrt{1 - 5\nu^2}\right) \quad .
\]
Thus, when the argument of the square root is negative, there are no real solutions, which means \( G(y) \) is monotonically increasing and \( G(y) = f_0^2 \) has a unique solution. This occurs for \( \nu^2 > \frac{1}{5} \).

For \( \nu^2 < \frac{1}{5} \), there are two solutions \( G'(y_\pm) = 0 \) with \( y_\pm > 0 \) and another two solutions at \( y = -y_\pm \), since \( G(y) \) is an odd function of \( y \). Note that \( G(y_-) > G(y_+) \). Thus, \( G(y) = f_0^2 \) has three solutions provided \( f_0^2 \in [G(y_+), G(y_-)] \cap [0, \infty) \). One then finds this is equivalent to the condition
\[
(3 + 2u)^{3/2} (1 - u) < \sqrt{\frac{3125}{312}} f_0^2 < (3 - 2u)^{3/2} (1 + u) \quad ,
\]
where \( u = \sqrt{1 - 5\nu^2} \in [0, 1] \). Note that for \( \nu^2 = \frac{1}{5} \) there root at \( f_0^2 = (2^9 \cdot 3^3 / 5)^{1/2} = 2.10325 \) is a double root. However, we still must check whether these solutions are stable. To do this, we compute the eigenvalues of the matrix \( M \), with
\[
M = \frac{1}{16} \begin{pmatrix}
8 - 5A^4 & 16\nu A \\
-16\nu A^{-1} & 8 - A^4
\end{pmatrix} \quad .
\]
The eigenvalues are \( \lambda_\pm = \frac{1}{2} T \pm \sqrt{\frac{1}{4} T^2 - D} \), where
\[
T = \text{Tr}(M) = 1 - \frac{3}{8} y^2 \quad , \quad D = \text{det} M = \frac{5}{500} y^4 - \frac{3}{16} y^2 + \nu^2 + \frac{1}{4} = \frac{1}{4} G'(y) \quad .
\]
The fixed point will be unstable if either of the eigenvalues has a positive real part. One possibility is a saddle point, which occurs for \( D < 0 \). This means \( G'(y) < 0 \), which means \( y \in [y_- , y_+] \). Thus, when we have three solutions, the middle one is always unstable.
Figure 1: Fixed point solutions corresponding to entrained phases of the forced modified van der Pol oscillator. Thin dashed curves correspond to different values of \( f_0 \).

The other possibility is \( T > 0 \), leading to an unstable spiral or unstable node. This is equivalent to \( y^2 < \frac{8}{3} \). Since a global analysis for large \( A \) shows the flow is inward, we conclude that the coupled ODEs for \( A \) and \( \psi \) must have a stable limit cycle in the portion of the \((\nu, y)\) plane corresponding to an unstable node or unstable spiral, \textit{i.e.} where \( y < \sqrt{\frac{8}{3}} \) and \( G'(y) > 0 \). The line \( y = \sqrt{\frac{8}{3}} \) intersects the curve \( D = 0 \) at \( \nu = \frac{1}{3} \). Thus, the phase diagram resembles that of Fig. 3.4 in the Lecture Notes. To find the range of \( f_0^2 \) over which there is hysteretic jumping between stable branches over some interval \( \nu \in [\nu_-, \nu_+] \), we set \( G(y) = G'(y) = 0 \) and eliminate \( y \) to obtain \( f_0^2 = \sqrt{\frac{256}{3125}} (3 + 2u)^{3/2}(1 - u) \). We then evaluate \( G(y) = f_0^2 \), for the same value of \( \nu \), at the point where \( \text{Tr}(M) = 0 \), \textit{i.e.} \( y = \sqrt{\frac{8}{3}} \), which yields \( f_0^2 = \frac{4}{3} \sqrt{\frac{8}{3}} (14 - 9u^2) \). Eliminating \( f_0^2 \), we arrive at the quintic equation

\[
125 (14 - 9u^2)^2 = 972 (3 + 2u)^3 (1 - u)^2 .
\]

The solution over the interval \( u \in [0, 1] \) is \( u = 0.350851 \), which gives us \( f_{0,\text{min}}^2 = 1.87136 \). Thus, hysteresis occurs for \( f_0^2 \in [1.87136, 2.10325] \), \textit{i.e.} \( f_0 \in [1.3680, 1.4503] \). Note one can also have hysteresis between a stable entrained solution and a stable limit cycle as parameters are varied.
Figure 2: Entrained and heterodyne behavior of the forced modified van der Pol oscillator, with $\epsilon = 0.1$ and $\nu = 0.4$.

(3) Consider shock formation in the inviscid Burgers’ equation, $c_t + c c_x = 0$. Let the function $c(\zeta) = c(x = \zeta, t = 0)$ be given by the triangular profile,

$$c(\zeta) = c_0 \left( \frac{a - |\zeta|}{a} \right) \Theta(a - |\zeta|).$$

(a) Find the break time $t_B$.

(b) Implement the shock fitting protocol and find $\zeta_-(t)$, $\zeta_+(t)$, and $x_s(t)$.

(c) Find the shock discontinuity $\Delta c(t)$ for $t > t_B$.

(d) Sketch $c(x, t)$ vs. $x$ for $t/t_B = 0, \frac{1}{2}, 1, 2,$ and $4$. Show that for $t \geq t_B$, $c(x, t)$ vs. $x$ has
the form of a right triangle whose area is given by \[ \int_{-a}^{a} \frac{d\zeta}{c(\zeta)}. \]

(e) Without shock fitting, sketch the characteristics in the \((x, t)\) plane and highlight the region where they cross. Then sketch the characteristics after shock fitting. Hint: Your sketches should roughly resemble those in Fig. 4.13 of the Lecture Notes.

Solution:

(a) The break time is

\[ t_B = \min_{c' < 0} \left( -\frac{1}{c'(\zeta)} \right) \equiv -\frac{1}{c'(\zeta_B)}. \]

Thus, \( t_B = a/c_0 \).

(b) We have two shock fitting equations:

\[ x_s = \zeta_+ + c_- t = \zeta_+ + c_+ t, \]

where \( c_\pm \equiv c(\zeta_\pm) \), and

\[ \frac{1}{2} (\zeta_+ - \zeta_-) (c_+ + c_-) = \int_{\zeta_-}^{\zeta_+} \frac{d\zeta}{c(\zeta)}. \]

Clearly \( \zeta_+ > a \) and therefore \( c_+ = 0 \). We also have \( \zeta_- < 0 \). The second of our shock fitting equations then gives

\[ \zeta_+ = \zeta_- + \frac{a}{a + \zeta_-} \left( a - 2\zeta_- - \frac{\zeta_-^2}{a} \right). \]

Figure 3: Left: Shock fitting requires the burgundy and green hatched areas to be equal. Right: Evolution of the initial profile at times \( \tau = t/t_B = 0 \) (black), \( \tau = \frac{1}{2} \) (blue), \( \tau = 1 \) (red), \( \tau = 2 \) (magenta), and \( \tau = 4 \) (green). The dashed line shows the shock discontinuity.
The first shock fitting equation gives $\zeta_+ - \zeta_- = c_- t$, and eliminating $\zeta_+$ yields

$$c_0 t = \left( \frac{a}{a + \zeta_-} \right)^2 \left( a - 2\zeta_- - \frac{\zeta_-^2}{a} \right).$$

At this point it is convenient to define the dimensionless time $\tau \equiv c_0 t/a = t/t_B$ as well as $q_\pm \equiv \zeta_\pm / a$. Note $q_b = x_b / a = q_+$ because $c_+ = 0$. Solving, we obtain

$$q_-(\tau) = -1 + \sqrt{\frac{2}{1 + \tau}}, \quad q_+(\tau) = -1 + \sqrt{2(1 + \tau)}.$$  

(c) The dimensionless velocity is $\bar{c} = c / c_0$. Note $\bar{c}_\pm = 1 - |q_\pm|$. The shock discontinuity is then

$$\Delta \bar{c}(\tau) = \sqrt{\frac{2}{1 + \tau}},$$

where $\tau \geq 1$. Note $\Delta \bar{c}(\tau = 1) = 1$, which is nongeneric, since the discontinuity usually grows from zero starting at the break time. The nongeneric nature here is due to the piecewise linear initial profile. Note also $\Delta \bar{c}(\tau) \propto \tau^{-1/2}$ as $\tau \rightarrow \infty$. See Fig. 3.

Figure 4: Top: Characteristics prior to shock fitting, showing intersection in the hatched region. Bottom: Characteristics with shock fitting. The shock trajectory is shown in red.
(d) For $t > t_B$, the curve $\bar{c}(q, \tau)$ is a right triangle whose base is $1 + q_+(\tau)$ and height is $\Delta \bar{c}(\tau)$. Thus, the dimensionless area is

$$A(\tau > 1) = \frac{1}{2} (1 + q_+(\tau)) \Delta \bar{c}(\tau) = 1 = \int_{-1}^{1} dq \ (1 - |q|) ,$$

and so the area is preserved. For $\tau < 1$, we have $\bar{c}(q, \tau)$ is a triangle connecting the points $(-1, 0)$ to $(\tau, 1)$ to $(1, 0)$, since the peak value moves with $\bar{c}_{\text{max}} = 1$. The area is again

$$A(\tau < 1) = \frac{1}{2} (1 + \tau) + \frac{1}{2} (1 - \tau) = 1 .$$

(e) See Fig. 4.