Consider a tri-junction connecting three identical one-dimensional leads. The leads are each connected to reservoirs described by chemical potentials $\mu_\alpha$ at a fixed temperature $T$. The $S$-matrix relates incoming and outgoing plane wave states, as usual:

$$
\begin{pmatrix}
A^{\text{OUT}} \\
B^{\text{OUT}} \\
C^{\text{OUT}}
\end{pmatrix} =
\begin{pmatrix}
r_A & t_{AB} & t_{AC} \\
t_{BA} & r_B & t_{BC} \\
t_{CA} & t_{CB} & r_C
\end{pmatrix}
\begin{pmatrix}
A^{\text{IN}} \\
B^{\text{IN}} \\
C^{\text{IN}}
\end{pmatrix}.
$$

(a) Following the arguments in §2.3 of the lecture notes, derive, mutatis mutandis, an equation relating the current $I_\alpha$ in terms of the chemical potentials $\mu_\alpha$ of the reservoirs. Be sure to comment on aspects such as current conservation.

Now consider the tight binding tri-junction model described in fig. 1. The hopping matrix elements along the chains are all identical and are equal to $t$. The hopping matrix elements on the internal triangle are all identical and equal to $t_\Delta$. The on-site energies for all sites are identical and equal to $\varepsilon_0 = 0$.

(b) Derive an expression for the $S$-matrix for this system. You should write

$$A_n = A^{\text{IN}} e^{-ikn} + A^{\text{OUT}} e^{ikn},$$

with corresponding expressions for the $B$ and $C$ leads. (See the hint at the end of the problem for some mathematical guidance.)

(c) Suppose $\mu_A = eV$ and $\mu_B = \mu_C = 0$. Derive an expression for the current $I_B$ at $T = 0$. Plot the dimensionless conductance $(h/e^2) \times (I_B/V)$ versus the dimensionless incident energy $\varepsilon = E/t$ over the allowed range $\varepsilon \in [-2, 2]$ for several values of the ratio $r \equiv t_\Delta/t$.

Hint: At some point, you may find it necessary to invert a matrix of the form

$$R = \begin{pmatrix}
a & b & b \\
b & a & b \\
b & b & a
\end{pmatrix}.$$

To this end, note that we can write

$$R = (a - b) \mathbb{I} + 3b |\psi\rangle \langle \psi|,$$

where $\bar{\psi}^r = \frac{1}{\sqrt{3}}(1, 1, 1)$, so $|\psi\rangle \langle \psi|$ is a matrix whose elements are all equal to $\frac{1}{3}$. But then

$$R = (a - b) Q_\psi + (a + 2b) P_\psi,$$

where $P_\psi = |\psi\rangle \langle \psi|$ is the projector onto $|\psi\rangle$, and $Q_\psi = \mathbb{I} - P_\psi$ is the projector onto the two-dimensional subspace orthogonal to $|\psi\rangle$. But then, clearly

$$R^{-1} = \frac{1}{a - b} Q_\psi + \frac{1}{a + 2b} P_\psi = \frac{1}{(a - b)(a + 2b)} \begin{pmatrix}
a + b & -b & -b \\
-b & a + b & -b \\
-b & -b & a + b
\end{pmatrix}.$$
Consider a spin-$S$ quantum Heisenberg model on a bipartite lattice. The A sublattice sites are located at positions $R$ and the B sublattice sites at $R + \delta$, where $R$ is an element of some Bravais lattice and $\delta$ is the sole basis vector. The Hamiltonian is

$$
H = - \sum_{R, R'} \left\{ \frac{1}{2} J_{\text{AA}}(|R - R'|) \mathbf{S}_A(R) \cdot \mathbf{S}_A(R') + \frac{1}{2} J_{\text{BB}}(|R - R'|) \mathbf{S}_B(R) \cdot \mathbf{S}_B(R') \\
+ J_{\text{AB}}(|R - R' - \delta|) \mathbf{S}_A(R) \cdot \mathbf{S}_B(R') \right\} - \gamma \sum_{R} \left\{ H_A(R) S_z^A(R) + H_B(R) S_z^B(R) \right\}
$$

where $\mathbf{S}_A(R)$ is the spin operator at the A sublattice site located at $R$, and $\mathbf{S}_B(R)$ is the spin operator at the B sublattice site located at $R + \delta$.

(a) Compute the susceptibility

$$
\chi_{AB}(q) = \frac{\partial M_A(q)}{\partial H_B(q)} \bigg|_{H_A=H_B=0}
$$

using a mean field approach. Recall the local susceptibility for a single Heisenberg spin is $\chi_0(T) = \gamma^2 p^2 / k_B T$, where $p^2 = \frac{1}{3} S(S+1)$. (You should express your answer in terms of $\chi_0$ and other relevant quantities.)

(b) Consider the model on a honeycomb lattice. The AB interactions are between nearest neighbors only, and are given by $J_{\text{NN}} < 0$ (antiferromagnetic). The AA interactions are between next-nearest neighbors only, and are given by $J_{\text{NNN}} > 0$ (ferromagnetic). Find an expression for $T_c$. 

Figure 1: A tri-junction formed from three semi-infinite single-orbital tight-binding chains.
Big hint: You should derive an equation of the form $R_{ab}(q) M_b(q) = H_a(q)$, where $a$ and $b$ run over sublattices and $R(q)$ is some matrix. The susceptibility matrix is the inverse of $R(q)$, and $\chi_{AB}(q)$ is the upper right element. To find $T_c$, set $\text{det}(R) = 0$.

(c) Consider a nearest-neighbor Heisenberg antiferromagnet on the honeycomb lattice with an easy axis anisotropy term. The Hamiltonian is

$$\mathcal{H} = J \sum_{\langle ij \rangle} \left( S_i^x S_j^x + S_i^y S_j^y + \Delta S_i^z S_j^z \right),$$

where $J > 0$ and $\Delta > 1$. Derive the spin wave spectrum. For $10^{50}$ quatroos extra credit, plot the spin wave dispersion on a triangle $\Gamma$–$K$–$M$–$\Gamma$ in the Brillouin zone.