example (from departmental exam)

Find steady state E-field and charge density inside sphere:

\[ 0 = E + \nabla \times B = -\nabla \Phi_m + (\mu_0 \sigma_c) \times \vec{B} \]

\[ 0 = -\nabla \Phi + \frac{\omega B \rho}{c} \]

\[ \Phi_m = \frac{\omega B \rho r^2}{c} + C \]

\[ -4\pi P = \frac{1}{\mu_0} \frac{\partial \Phi_m}{\partial r} = \frac{2\omega B}{c} \]
returning to spherical coordinates
\[ r_1^2 = r^2 \sin^2 \theta = r^2 (1 - \cos^2 \theta) \]

\[ \Phi_{\text{in}} = \frac{\omega B}{2c} \frac{r^2}{2} (1 - \cos^2 \theta) + C \]

\[ \Phi_{\text{out}} = \sum \left( A_l r^l + B_l r^{l+1} \right) P_l (\cos \theta) \]

\[ \Phi (r) \to 0 \quad \text{as} \quad r \to \infty \]

implies that \( A_p = 0 \) for all \( l \)

\[ \Phi_{\text{in}} (a, \theta) = \Phi_{\text{out}} (a, \theta) \quad \text{E} \text{t} \text{c} \text{ont}. \]

\[ C + \frac{\omega B}{2c} a^2 (1 - \cos^2 \theta) = \frac{B_0}{a} P_0 (\cos \theta) + \frac{B_1}{a^2} P_1 (\cos \theta) + \frac{B_2}{a^3} P_2 (\cos \theta) + \ldots \]

\[ + \frac{B_3}{a^3} \left[ \frac{3}{2} \cos^2 \theta - 1 \right] \]
\[
1 - \cos^2 \theta = -\frac{2}{3} P_2(\cos \theta) - \frac{1}{3} \sqrt{3} \cos \theta
\]

\[
\mathcal{E} + \frac{\omega B}{2c} a^2 \left[ -\frac{2}{3} P_2(\cos \theta) + \frac{2}{3} \right] = \frac{B_0}{a} P_0(\cos \theta) + \frac{B_1}{a} P_1(\cos \theta)
\]

\[
+ \frac{B_2}{a^3} P_2(\cos \theta) + \ldots
\]

\[
\therefore B_2 = -\frac{2}{3} \frac{\omega B a^5}{2c}
\]

\[
B_1 = 0
\]

\[
\mathcal{E} + \frac{\omega B a^2}{2c} \frac{2}{3} = 0
\]

\[
\Phi_{oa}(\text{sol}) = -\frac{\omega B a^5}{8c} \frac{r^2}{r^3} \frac{P_2(\cos \theta)}{2c}
\]

quadrapole field outside
surface charge

\[
4\pi \sigma = \frac{2\Phi_\text{in}}{\partial r} - \frac{2\Phi_\text{out}}{\partial r} \quad r = a
\]

\[
= + \frac{\omega B}{c} \frac{q^2}{3} P_2 [\cos \theta] + \frac{\omega B}{c} P_2 [\cos \theta]
\]

\[
= - \frac{\omega B}{3c} P_2 [\cos \theta] + \frac{2\omega B}{3c}
\]

Have neglected change in B due to rotating charge (current)

Homework:

12. A superconducting sphere excludes magnetic field from the interior (Type 1 superconductor). Calculate the magnetic field near the sphere if it is placed in a uniform magnetic field \( B_0 \).
(13) Everywhere inside a solid sphere of radius \( R \), the magnitude of the magnetization is \( M = \text{const.} \). Calculate \( \mathbf{B} \) in \( \mathbb{R}^3 \) of a sphere with \( \nabla \times \mathbf{H} = 4\pi \mathbf{M} = 0 \).

\[ \therefore \mathbf{H} = -\nabla \varphi \]

\[ 0 = \nabla \cdot \mathbf{B} = \nabla \cdot (-\nabla \varphi + 4\pi \mathbf{M}) \]

\( \varphi \) is non-zero only at surface

Multipole expansions re-examined

Jackson, Chapter 4

\[
\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{\mathbf{e}_l}{r'} Y_l^m (\theta, \phi') Y_l^m (\theta, \phi)
\]

\( \mathbf{e}_l \) is smaller of \( |l|, |l'| \)

\( r \) is larger of \( |r|, |r'| \)
potential at large distance from localized charge distribution:

\[ \Phi(r) = \frac{\rho(r)}{4\pi r} \]

\[ \Phi(r) = \sum_{l,m} \frac{4\pi}{2l+1} \left[ \int \frac{r}{r} \, d^3r \right] Y_{lm}(\theta, \phi) \left( \frac{1}{r} \right) \]

\[ q_{0,0} = \frac{1}{\sqrt{4\pi}} \int \rho(r) \, d^3r \]

\[ q_{10} = \sqrt{\frac{3}{4\pi}} \int \frac{\rho(r) \cos \theta \, d^3r}{2} \]

\[ q_{11} = -\sqrt{\frac{3}{8\pi}} \int \rho(r) \, d^3r \]

\[ q_{20} = \frac{1}{\sqrt{4\pi}} \int \frac{\rho(r) \cos \theta \, d^3r}{\frac{3}{2} \frac{r^2 - z^2}{r}} \]

\[ \Phi_{ij} = \int \left[ 3r_i \cdot r_j - r_i^2 \delta_{ij} \right] \rho(r) \]
Electromagnetic waves  Chapter 7
L + L Vol 12 chapter 5
L + L Vol 18 chapter 9

In vacuum

\[ \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

\[ \nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^3} \frac{\partial \mathbf{B}}{\partial t} \]
\[ \nabla \times \mathbf{B} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

\[ \nabla \cdot \nabla \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

\[ \Box \mathbf{E} \cdot \Box \mathbf{E} = 0 \]

Monochromatic plane wave

\[ \mathbf{E} = R_0 \mathbf{E}_0 e^{-i\mathbf{k} \cdot \mathbf{r} - i\omega t} \]
\[ \omega^2 = c^2 k^2 \]

\[ \mathbf{B} = \nabla \times \mathbf{E} = R_0 i \mathbf{k} \cdot \mathbf{E}_0 e^{-i\mathbf{k} \cdot \mathbf{r} - i\omega t} \]
\[ \therefore \mathbf{k} \cdot \mathbf{E}_0 = 0 \] transverse wave
\[
\text{Let } B = R e \hat{B}_0 e^{i \kappa L - i \omega t} \quad \uparrow \text{this function's form is necessary}
\]

\[
R e \frac{i \kappa \times E_0}{\omega} e^{i \kappa L - i \omega t} = R e \frac{i \kappa \omega \hat{B}_0}{\omega} e^{i \kappa L - i \omega t} 
\]

\[
\therefore \quad \hat{B}_0 = \frac{e^{i \kappa L}}{\omega} \frac{i \kappa}{\omega} \frac{\omega}{\omega} = \vec{k} \times E_0
\]

\[
\begin{align*}
\hat{E}_0 &= \uparrow \text{necessary since electric and magnetic field energy are traded back and forth} \\
|\hat{E}_0| &= |\hat{B}_0| \\
\end{align*}
\]
\textbf{Polarization}

Let \( \mathbf{k} = k \hat{\mathbf{z}} \)

\[
\mathbf{E} = R_0 \mathbf{E}_0 \ e^{i(k_z z - \omega t)}
\]

\[ Z \mathbf{E}_0 = 0 \]

\( \uparrow \text{complex vector} \)

\[
E_0^+E_0 = E_0^2 = |E_0^2| e^{-2i\phi}
\]

Let \( \mathbf{b} = E_0 e^{+i\phi} \)

\[
\Rightarrow b^2 = E_0^2 e^{+2i\phi} = |E_0^2|
\]

\( \uparrow \text{real} \)

Let \( \mathbf{b} = b_1 + i b_2 \)

\( \Rightarrow \text{real} \)

\[
b^2 = b_1^2 - b_2^2 - 2i b_1 \cdot b_2
\]

\[ b_1 \cdot b_2 = 0 \]
\begin{align*}
\text{Let } \mathbf{X} &= \frac{\mathbf{v}_1}{|\mathbf{v}_1|} \times \frac{\mathbf{v}_2}{|\mathbf{v}_2|}, \quad \mathbf{v}_1 \mathbf{v}_2 = 1 \\
\therefore \quad \mathbf{E}_0 &= (b_1 \mathbf{x} - i b_2 \mathbf{y}) e^{-i\alpha} \\
\text{where positive } \pm \text{ can be negative} \\
\mathbf{E}_x &= b_1 \cos(kz - \omega t + \phi) \\
\mathbf{E}_y &= b_2 \sin(kz - \omega t + \phi) \\
\therefore \quad \frac{\mathbf{E}_x^2}{b_1^2} + \frac{\mathbf{E}_y^2}{b_2^2} &= \cos^2(\phi) + \sin^2(\phi) = 1 \\
\uparrow \quad \text{equation for ellipse} \\
\therefore \quad \text{elliptically polarized}
\end{align*}
Special cases

1. Circularly polarized for $|b_1| = |b_2|$

$$E_0 = b_1 e^{i \alpha} (\hat{x} + i \hat{y})$$

+ sign: left hand circularly polarized (positive helicity)

- sign: right hand circularly polarized (negative helicity)

$$\hat{E}_y = \frac{b_1}{\sqrt{2}} \sin(2\omega t + \phi)$$

fixed point in space

fixed point in time

Homework

(14) J 7.28 (plane wave of finite cross section)

(15) J 7.29 $\frac{1}{\sqrt{w}} = \pm \frac{1}{c_0}$
(2) linearly polarized for

\[ b_1 \circ b_2 = 0 \]

\[ E_0 = b_1 e^{-i\alpha} \]

\[ E = \hat{x} b_1 \cos(kt - \omega t - \alpha) \]

Note that superposition of left hand circularly polarized waves and right hand circularly polarized waves of equal amplitude make a linearly polarized wave.
Time average quantities

\[ g(t) = R_0 \; g \; e^{-i \omega t} \]

\[ h(t) = R_0 \; h \; e^{-i \omega t} \]

\[ \langle g(t) h(t) \rangle = \frac{1}{2} \int_0^{2 \pi} dt \; [ g_r \cos \omega t + g_i \sin \omega t ] \cdot [ h_r \cos \omega t + h_i \sin \omega t ] \]

\[ = \frac{1}{2} [ g_r h_r + g_i h_i ] = \frac{1}{2} R_0 g \; R_0 h^* \]

Time average Poynting flux

\[ \mathbf{S} = R_0 \mathbf{E}_0 \; e^{-i \omega t} \]

\[ \mathbf{S} = R_0 \mathbf{B}_0 \; e^{i \omega t} \]

\[ \mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0 \]
\[ <S> = \frac{1}{2} R_0 C \frac{E_0 \times B_0}{4\pi} \]
\[ <S> = \frac{C}{8\pi} \frac{E_0^2 + E_0 \cdot E_0}{k} \]
\[ \left| \vec{E} \right|^2 = \left| \vec{E}_0 \right|^2 \]
\[ \text{time average energy density} \]
\[ <\hat{W}> = \frac{1}{2} R_0 \frac{E_0 \times B_0}{4\pi c} \approx \frac{E_0^2}{8\pi} \]
\[ \text{time average momentum density} \]
\[ <\vec{q}> = \frac{1}{2} R_0 \frac{E_0 \times B_0}{4\pi c} = \frac{\left| \vec{E}_0 \right|^2 R_0}{8\pi c} \]
\[ \text{note that } S = W c \hat{\mathbf{e}} \]
\[ \text{note that} \]
\[ <\vec{W}> = \frac{1}{2} R_0 \frac{E_0 \times B_0}{4\pi c} \]
\[ \text{(i.e., } \vec{E} = c\hat{\mathbf{e}} \text{ for photons)} \]
Plane wave incident on conducting surface

Case 1, polarization \( \perp \) to plane of incidence

\[
E(t) = R_0 E_0 e^{i(kr - \omega t)}
\]

\[
B_0 = \hat{e} \times E_0
\]

Reflected wave

\[
E'(t) = R_0 E_0 e^{i(k'r - \omega t)}
\]

\[
B'_0 = \hat{e} \times E'_0
\]
\[-\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} \Rightarrow E_t \text{ to cont.}; \quad E_t = 0\]
\[\nabla \cdot B = 0 \Rightarrow B_n \text{ to cont.}; \quad B_n = 0\]

Note that

\(E_t\) and \(B_n\) are not continuous.

Matching requires

\[(x, 0) - \omega t = \left(\frac{x}{v}, 0\right) - \omega t + \text{ cons}.\]
\[y = 0 \quad y = 0\]

\[w = w' \Rightarrow \frac{k_x}{v} = \frac{k_x}{v}\]
\[k_y = k_y, \quad \frac{k_z}{v} = \frac{k_z}{v}\]

Diff. thus case

\[\theta = \theta'\]
at the surface \( (E + E')_t = 0 \)
\[
(\mathbf{B} + \mathbf{B'})_n = 0
\]
satisfied if \( E' = -E \)

Case 2: Polarization parallel to plane of incidence

\[
(E + E')_t = 0
\]
\[
(\mathbf{B} + \mathbf{B'})_n = 0
\]
satisfied if \( |E| = |E'| \)
A plane wave $E = E_0 e^{ik \cdot r - i\omega t}$ is incident at an angle $\theta$ on a conducting surface.

(a) Using the stress tensor, calculate the average force per area (radiative pressure) on the surface.

(b) Also calculate this quantity using conservation of momentum.

A plane wave is normally incident on a conducting surface that moves with velocity $v$. Calculate the radiative pressure as measured by an observer on moving.
\textbf{Dispersion}

In general $\varepsilon$ and $\mu$ depend on frequency:

\[ \varepsilon(\omega) = \varepsilon(\omega) \quad \mu(\omega) = \mu(\omega) H(\omega) \]

For monochromatic plane wave:

\[ E = \Re E_0 e^{ik \cdot r} e^{-i\omega t} \quad B = \Re B_0 e^{ik \cdot r} e^{-i\omega t} \]

\[ i\kappa \times E_0 = \frac{\omega}{c} B_0 \quad i\kappa \cdot (\varepsilon(\omega) E_0) = 0 \]

\[ i\kappa \times (\mu(\omega)^{-1} B_0) = -\frac{\omega}{c} \varepsilon(\omega) E_0 \quad \kappa \cdot B_0 = 0 \]

\[ i\kappa \times \kappa \times E_0 = -\frac{\omega^2}{c^2} \varepsilon(\omega) \mu(\omega) E_0 \]

\[ \kappa^2 \varepsilon_0 = \frac{\omega^2}{c^2} \varepsilon(\omega) \mu(\omega) \]

\[ \omega = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \text{ rad} \text{ s}^{-1} \]
We say that there is dispersion when $w$ and $k$ are not linearly related, wave packet distorts as it propagates.

**Phase velocity**

\[ v_{ph} = \frac{\omega}{k} \]

for \( \omega = \omega_0 + v_{ph} t \)

\[ k \cdot r - \omega t = k \cdot r_0 + k v_{ph} t = c t = \text{const.} \]

**Index of refraction** for transparent medium (\( k_0, \sqrt{\epsilon_0(\omega)} \mu(\omega) \)) is real at freq. of interest.

\[ n(\omega) = \frac{c}{k_0 \mu(\omega)} = \sqrt{\epsilon_0(\omega)} \mu(\omega) \]
Reflection and refraction at an interface between two transparent media.

Case 1: Polarization perpendicular to plane of incidence.

\[
B = \frac{c}{\omega} k \times E
\]

Matching requires:

\[
k_x' = k_x = k_x'' \quad , \quad k_z' = k_z'' = k_z = 0
\]

\[
k \sin \theta = k' \sin \theta' = k'' \sin \theta''
\]

\[
\omega = \omega' = \omega''
\]

\[
n' = \frac{C_1}{n_1} \quad , \quad n = \frac{C_1}{n_1} \quad , \quad n'' = \frac{C_1}{n_1}
\]
Snells' Law: 
\[ \sin \theta = \sin \theta'' \quad \text{or} \quad \theta = \theta'' \]

Start with \( b.c. \),

1. \( D_n \) is cont. (satisfied)

2. \( E_t \) is cont.
\[ E + E'' = E' \]

3. \( B_n \) is cont.
\[ B \sin \theta + B'' \sin \theta'' = B' \sin \theta' \]
\[ \frac{\partial B}{\partial t} = \frac{\partial B''}{\partial t''} = \frac{\partial B'}{\partial t'} \]
\[ k \sin \theta = k'' \sin \theta'' = k' \sin \theta' \]
\[ w = w_0 \]
\[ \therefore E - E'' = E' \] same as (2)

4. \( H_t \) is cont.
\[ \frac{1}{\mu} \left[ \frac{\partial B}{\partial t} \cos \theta + \frac{\partial B}{\partial t''} \cos \theta'' \right] = -\frac{1}{\mu} \frac{c_k' E' \cos \theta'}{w} \]
\[ \sqrt{\frac{\mu}{\mu_1}} (E - E'') \cos \theta = \frac{\mu}{\mu_1} E' \cos \theta' \]
\[ E - E'' = \sqrt{\frac{m_2}{m_1}} E' \frac{\cos\theta}{\cos\theta} = \frac{n_2}{n_1} \frac{u_1}{u_2} \frac{\cos\theta}{\cos\theta} E' \]

\[ E + E'' = E' \]

\[ 2E = E' \left[ 1 + \frac{n_2}{n_1} \frac{u_1}{u_2} \frac{1 - \left( \frac{m_1}{m_2} \right)^2 \sin^2\theta}{\cos\theta} \right] \]

\[ \frac{E'}{E} = \frac{2m_1 \cos\theta}{n_1 \cos\theta + \frac{\sqrt{n_1^2 - n_2^2 \sin^2\theta}}{u_2}} \]

\[ \frac{E''}{E} = \frac{n_1 \cos\theta - \frac{u_1}{u_2} \frac{\sqrt{n_1^2 - n_2^2 \sin^2\theta}}{u_2}}{n_1 \cos\theta + \frac{u_1}{u_2} \frac{\sqrt{n_1^2 - n_2^2 \sin^2\theta}}{u_2}} \]

Note that \( \frac{u_1}{u_2} \) is typically near 1.
Case 2: Polarization \( \perp \) to plane of incidence

\[ E' = \frac{2 \pi n_2 \cos \theta}{\mu_2^2 \cos \theta + n_2 \sqrt{n_2^2 - n_1^2 \sin^2 \theta}} \]

\[ E'' = \frac{-\mu_1 n_2^2 \cos \theta - n_1 \sqrt{n_2^2 - n_1^2 \sin^2 \theta}}{\mu_2^2 \cos \theta + n_2 \sqrt{n_2^2 - n_1^2 \sin^2 \theta}} \]

Typically \( \frac{n_1}{n_2} \approx 1 \)